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Necessary and sufficient conditions for linear suboptimality in constrained optimization

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Abstract This article is devoted to the study of some extremality and optimality notions that are different from conventional concepts of optimal solutions to optimization-related problems. These notions reflect certain amounts of linear subextremality for set systems and linear suboptimality for feasible solutions to multiobjective and scalar optimization problems. In contrast to standard notions of optimality, it is possible to derive necessary and sufficient conditions for linear subextremality and suboptimality in general nonconvex settings, which is done in this article via robust generalized differential constructions of variational analysis in finite-dimensional and infinite-dimensional spaces.

Keywords Non-smooth optimization · Variational analysis · Generalized differentiation · Linear subextremality and suboptimality · Necessary and sufficient conditions

Mathematics Subject Classification (2000) 90C30 · 49J52 · 49J53

1 Introduction

It has been well recognized that, excepts convex programming and related problems with a convex structure, *necessary* conditions are usually *not sufficient* for conventional notions of optimality. Observe also that major necessary optimality conditions in all the branches of the classical and modern optimization theory (e.g., Lagrange multipliers and Karush–Kuhn–Tucker conditions in non-linear programming, the Euler–Lagrange equation in the calculus of variations, the Pontryagin maximum principle in optimal control, etc.) are expressed in *dual* forms involving *adjoint* variables. At the same time, the very *notions of optimality*, in both scalar and vector frameworks, are formulated of course in *primal* terms.

A challenging question is to find certain modified notions of local optimality so that *first-order* necessary conditions known for the previously recognized notions become *necessary*

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and sufficient in the new framework. Such a study has been started by Kruger (see [7,8] and the references therein), where the corresponding notions are called "weak stationarity." It seems that the main difference between the conventional notions and those studied in [7,8] and in this article is that the latter relate to a certain (sub)optimality not at the point in question but in a *neighborhood* of it, and that they involve a *linear rate* in the sense precisely defined below. To some extent, this is similar to the linear rate in *Lipschitz continuity* (in contrast merely to continuity) as well as in modern concepts of *metric regularity* and *linear openness*, which distinguishes them from the classical regularity and openness notions of non-linear analysis. On this basis we suggested in [13] to use the names "linear subextremality" for set systems and "linear suboptimality" for the corresponding notions in optimization problems.

As has been fully recognized just in the framework of modern variational analysis (even regarding the classical settings), the *linear rate nature* of the fundamental properties involving Lipschitz continuity, metric regularity, and openness for single-valued and set-valued mappings is the *key issue* allowing us to derive *complete characterizations* of these properties via appropriate tools of generalized differentiation; see the books [12,17] and their references. Precisely the same linear rate essence of the (sub)extremality and (sub)optimality concepts studied in this article is the driving force ensuring the possibility to justify the validity of known necessary extremality and optimality conditions for the conventional notions as *necessary and sufficient* conditions for the new notions under consideration.

In contrast to [7,8], where dual criteria for "weak stationarity" are obtained in "fuzzy" forms involving Fréchet-like constructions at points *nearby* the reference ones, in this paper (cf. also [13, Chapt. 5]) we pay the main attention to *pointwise* conditions expressed via the basic *robust* generalized differential constructions of [12,13] *exactly at* the points in question. Besides the latter being more convenient for applications, we can significantly gain from such pointwise characterizations due to the well-developed (full) *calculus* enjoyed by the robust constructions, which particularly allows us to cover problems with various *constrained* structures important for both the optimization theory and its applications.

Observe that, from the viewpoint of deriving *necessary and sufficient* conditions for linear suboptimality, we need calculus rules of not merely the (right) *inclusion type* as required by the majority of applications, but largely of the *equality type*, which are available as well [12] for our basic generalized differential constructions way beyond convexity. Furthermore, in *infinite-dimensional* spaces one also needs calculus of the so-called *sequential normal compactness* (SNC) properties (automatic in finite dimensions), which is strongly developed in the book [12]. Based on these calculi, we obtain characterizations and verifiable necessary conditions for linear suboptimality in various classes of structured optimization-related problems, including those known as *mathematical programs with equilibrium constraints* (MPECs) and *equilibrium problems with equilibrium constraints* (EPECs).

The rest of the article is organized as follows. In Sect. 2 we present preliminaries from variational analysis and generalized differentiation widely used in this work. Section 3 deals with geometric aspects of linear suboptimality concerning *linear subextremality* (or subextremality at a linear rate) for systems of sets. We show that the relations of the *exact extremal principle* [12] are *necessary* for linear subextremality under general assumptions in Asplund spaces being also *sufficient* for this property in finite dimensions.

Section 4 is devoted to the study of linear suboptimality in general problems of constrained *multiobjective optimization*. We derive pointwise necessary as well as necessary and sufficient conditions for linear suboptimality in both finite and infinite dimensions. The results are specified for problems with operator and functional constraints and also applied to general EPECs treated from the viewpoint of multiobjective optimization.

The last Sect. 5 of the paper concerns the study of linear suboptimality in constrained *minimization* problems. We obtain pointwise necessary conditions and characterizing results for the corresponding notion of *linear subminimality* and apply them to general MPECs and some specifications particularly governed by variational inequalities and their generalizations.

Our notation is basically standard; cf. [12,17]. In particular, $I\!B$ stands for the unit closed ball of the space in question, while $B_r(x)$ signifies the ball centered at x with radius $r > 0$. As usual, $I\!N := \{1, 2, \ldots\}$. Given a set-valued mapping $F\colon X \rightrightarrows X^*$ between a Banach space X and its topological dual X^*, denote by

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* \Big| \ \exists \ \text{sequences} \ x_k \to \bar{x} \ \text{and} \ x_k^* \xrightarrow{w^*} x^* \\ \text{with} \ x_k^* \in F(x_k) \ \text{for all} \ k \in I\!N\Big\} \tag{1}$$

the *sequential* Painlevé-Kuratowski upper/outer limit of F as $x \to \bar{x}$ with respect to the norm topology of X and the weak* topology w^* of X^*.

2 Preliminaries

We first recall the generalized differential constructions of variational analysis used in what follows; see the book [12] with the references and discussions therein and also [3,13,17] for some related and additional material.

Given a non-empty subset Ω of a Banach space X and a point $\bar{x} \in \Omega$, the (basic, limiting) *normal cone* to Ω at $\bar{x}$ is

$$N(\bar{x}; \Omega) := \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\Omega} \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x; \Omega), \tag{2}$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$, and where

$$\widehat{N}_\varepsilon(x; \Omega) := \Big\{x^* \in X^* \Big| \ \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \le \varepsilon\Big\} \tag{3}$$

is the set of ε-*normals* to Ω at $x \in \Omega$. When $\varepsilon = 0$ in (3), $\widehat{N}(x; \Omega) := \widehat{N}_0(x; \Omega)$ is a convex cone called the *prenormal cone* or the *Fréchet normal cone* to Ω at x. We can equivalently put $\varepsilon = 0$ in (2) if Ω is locally closed around $\bar{x}$ and the space X is *Asplund*, i.e., each separable subspace of X has a separable dual. The latter class includes all spaces with a Fréchet differentiable renorm, particularly every reflexive space. On the other hand, there are Asplund spaces that fail to have even a Gâteaux differentiable renorm; see [5,12] for more details, discussions, and references.

In contrast to (3), the normal cone (2) is often *nonconvex* enjoying nevertheless *full calculus* in the framework of Asplund spaces, while a number of useful calculus results are also available in arbitrary Banach spaces (see [12, Chapter 1–3]). This calculus is mainly based on *extremal/variational principles* that replace convex separation theorems in nonconvex settings. Accordingly, similar well-developed calculi hold true for the associated subdifferential and coderivative constructions concerning extended-real-valued functions and set-valued mappings defined below.

A set $\Omega \subset X$ is *normally regular* at $\bar{x} \in \Omega$ if

$$N(\bar{x}; \Omega) = \widehat{N}(\bar{x}; \Omega). \tag{4}$$

Besides convex sets, this property is satisfied in other important settings, particularly for sets described by smooth equalities and inequalities under the Mangasarian-Fromovitz constraint qualification. The reader can find more information about (4) and other notions of set regularity in [4,12,17] and the references therein. Note however that the normal regularity (4) *fails* for sets homeomorphic to *graphs* of single-valued nonsmooth Lipschitzian mappings, which is particularly the case of maximal monotone operators; see [12, Subsects. 1.2.2 and 3.2.4].

Considering next a set-valued mapping $F\colon X \rightrightarrows Y$ between Banach spaces and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ from its graph

$$\operatorname{gph} F := \big\{(x, y) \in X \times Y \big|\ y \in F(x)\big\},$$

we define the *normal coderivative* $D^*_N F(\bar{x}, \bar{y})\colon Y^* \rightrightarrows X^*$ of F at $(\bar{x}, \bar{y})$ by

$$D^*_N F(\bar{x}, \bar{y})(y^*) := \big\{x^* \in X^* \big|\ (x^*, -y^*) \in N\big((\bar{x}, \bar{y}); \operatorname{gph} F\big)\big\}. \tag{5}$$

The *mixed coderivative* of F at $(\bar{x}, \bar{y})$ is given by

$$D^*_M F(\bar{x}, \bar{y})(y^*) := \Big\{x^* \in X^* \Big| \exists\ \varepsilon_k \downarrow 0,\ (x_k, y_k) \overset{\operatorname{gph} F}{\to} (\bar{x}, \bar{y}),\ x^*_k \overset{w^*}{\to} x^*,$$
$$\|y^*_k - y^*\| \to 0 \text{ with } (x^*_k, -y^*_k) \in \widehat{N}_{\varepsilon_k}\big((x_k, y_k); \operatorname{gph} F\big) \text{ as } k \in I\!\!N\Big\}. \tag{6}$$

By (2) for $\Omega = \operatorname{gph} F \in X \times Y$, observe that the only difference between the normal and mixed coderivatives is that the norm convergence $y^*_k \to y^*$ is used in (6) instead of the weak* one $y^*_k \overset{w^*}{\to} y^*$ in the equivalent representation of (5). Note that we can put $\varepsilon_k \equiv 0$ in (6) and similarly in (5) if both spaces X and Y are Asplund while the graph of F is locally closed around $(\bar{x}, \bar{y})$. Clearly

$$D^*_M F(\bar{x}, \bar{y})(y^*) \subset D^*_N F(\bar{x}, \bar{y})(y^*) \ \text{ for all } y^* \in Y^*, \tag{7}$$

where the equality is obvious when Y is finite-dimensional. Generally, the case of equality in (7) is postulated in [12] as *strong coderivative normality* of F at $(\bar{x}, \bar{y})$. Some sufficient conditions for this important property of set-valued and single-valued mappings are listed in [12, Proposition 4.9]. They particularly include mappings that are *N-regular* at $(\bar{x}, \bar{y})$ (i.e., those whose graphs are normally regular (4) at $(\bar{x}, \bar{y})$; hence both convex-graph and strictly differentiable ones), also the so-called “strictly Lipschitzian mappings,” etc.

Another type of the mixed coderivative is defined in [12], under the name of the *reversed mixed coderivative* of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, by

$$\widetilde{D}^* F_M(\bar{x}, \bar{y})(y^*) := \big\{x^* \in X^* \big|\ -y^* \in D^*_M F^{-1}(\bar{y}, \bar{x})(x^*)\big\} \tag{8}$$

via the mixed coderivative (6) of the inverse mapping F^{-1}. It corresponds to the limiting construction in (6) with the reversed convergence $y^*_k \overset{w^*}{\to} y^*$ and $\|x^*_k - x^*\| \to 0$. The reversed mixed coderivative (8) clearly reduces to the normal one (5) when $\dim X < \infty$. Furthermore, we have

$$D^*_N f(\bar{x})(y^*) = D^*_M f(\bar{x})(y^*) = \widetilde{D}^*_M f(\bar{x})(y^*) = \big\{\nabla f(\bar{x})^* y^*\big\} \quad \text{for all } y^* \in Y^* \tag{9}$$

in any Banach spaces, provided that $F := f\colon X \to Y$ is single-valued and *strictly differentiable* at $\bar{x}$ (in particular, when it is continuously differentiable around this point).

Consider now an extended-real-valued function $\varphi\colon X \to \overline{I\!\!R} := [-\infty, \infty]$ finite at $\bar{x}$ and the associated *epigraphical multifunction* $E_\varphi\colon X \rightrightarrows I\!\!R$ given by

$$E_\varphi(x) := \big\{\mu \in I\!\!R \big|\ \mu \geq \varphi(x)\big\} \quad \text{with } \operatorname{gph} E_\varphi = \operatorname{epi} \varphi.$$

Then the *basic subdifferential* $\partial\varphi(\bar{x})$ and the *singular subdifferential* $\partial^{\infty}\varphi(\bar{x})$ of φ at $\bar{x}$ can be defined via the coderivative (5) of E_{φ} (which agrees with (6) in this case) by, respectively,

$$\partial\varphi(\bar{x}) := D_N^* E_{\varphi}(\bar{x}, \varphi(\bar{x}))(1) \;\text{ and }\; \partial^{\infty}\varphi(\bar{x}) := D_N^* E_{\varphi}(\bar{x}, \varphi(\bar{x}))(0). \tag{10}$$

If the space X is Asplund and if φ is lower semicontinuous (l.s.c.) around $\bar{x}$, then one has the analytic representation of both constructions in (10) by

$$\partial\varphi(\bar{x}) = \mathop{\mathrm{Lim\,sup}}_{x \xrightarrow{\varphi} \bar{x}} \widehat{\partial}\varphi(x), \quad \partial^{\infty}\varphi(\bar{x}) = \mathop{\mathrm{Lim\,sup}}_{\substack{x \xrightarrow{\varphi} \bar{x} \\ \lambda \downarrow 0}} \lambda\widehat{\partial}\varphi(x) \tag{11}$$

via the so-called *Fréchet subdifferential*

$$\widehat{\partial}\varphi(\bar{x}) := \left\{ x^* \in X^* \middle| \; \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \ge 0 \right\},$$

which is also known as the subdifferential in the sense of *viscosity solutions* of φ at $\bar{x}$. The symbol $x \xrightarrow{\varphi} \bar{x}$ in (11) signifies that $x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$. Recall that $\varphi\colon X \to \overline{I\!R}$ is *lower regular* at $\bar{x}$ if

$$\partial\varphi(\bar{x}) = \widehat{\partial}\varphi(\bar{x}),$$

which is the case of many important classes of functions (besides convex and smooth ones) encountered in optimization and variational analysis; see [12,17] for more details.

Finally in this section, recall certain *normal compactness* properties used in what follows. These properties automatically hold in finite dimensions being among the most essential ingredients of infinite-dimensional variational analysis and generalized differentiation. They are unavoidably present in calculus rules for robust generalized differential constructions discussed above and in the corresponding optimality conditions. It is important to emphasize that a well-developed *full calculus* is available for such properties (mostly in Asplund while also in Banach spaces), in the sense that they are known to be preserved while various operations are performed on sets, set-valued mappings, and extended-real-valued functions under natural qualification conditions; see [12] for more details.

Given a set $\Omega \subset X$, we say that it is *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if for any sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* \xrightarrow{w^*} 0$ one has

$$\|x_k^*\| \to 0 \;\text{ provided that }\; x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega) \quad \text{as } k \to \infty,$$

where ε_k can be equivalently omitted ($\varepsilon_k \equiv 0$) when X is Asplund and Ω is locally closed around $\bar{x}$. It is automatic when Ω is *compactly epi-Lipschitzian* (CEL) around $\bar{x}$ in the sense of Borwein and Strójwas [2], while in general the SNC requirement may be essentially weaker than the CEL one; see [6] for various examples in Banach and Asplund spaces.

Accordingly, a set-valued mapping $F\colon X \rightrightarrows Y$ is *SNC* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if its graph is SNC at this point. For mappings, a less restrictive property is important for applications. Namely, $F\colon X \rightrightarrows Y$ is *partially SNC* (PSNC) at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if $\|x_k^*\| \to 0$ whenever

$$x_k^* \in \widehat{N}_{\varepsilon_k}\big((x_k, y_k); \operatorname{gph} F\big), \;\; (x_k, y_k) \xrightarrow{\operatorname{gph} F} (\bar{x}, \bar{y}), \;\; x_k^* \xrightarrow{w^*} 0, \;\; \text{and } \; \|y_k^*\| \to 0,$$

with $\varepsilon_k \equiv 0$ for closed-graph mappings between Asplund spaces. We mention that F is automatically PSNC at $(\bar{x}, \bar{y})$ if it has Aubin's *Lipschitz-like* ("pseudo-Lipschitzian" [1]) property around this point.

3 Linear subextremality via the exact extremal principle

Following the *geometric approach* to variational analysis and generalized differentiation [12,13], we start with extremal properties of sets and then proceed with solutions to constrained optimization problems. Given two subsets Ω_1 and Ω_2 of a normed space X, recall [9] that $\bar{x} \in \Omega_1 \cap \Omega_2$ is a *local extremal point* of the set system $\{\Omega_1, \Omega_2\}$ if there exists a neighborhood U of $\bar{x}$ such that for any $\varepsilon > 0$ there is $a \in \varepsilon I\!B$ with

$$(\Omega_1 + a) \cap \Omega_2 \cap U = \emptyset.$$

Loosely speaking, the local extremality of sets at a common point means that they can be locally "pushed apart" by a small perturbation (translation) of one of them.

It is clear that every *boundary point* $\bar{x}$ of a closed set Ω is a local extremal point of the pair $\big\{\Omega, \{\bar{x}\}\big\}$. In general, this geometric concept of extremality covers conventional notions of optimal solutions to various problems of scalar and vector/multiobjective optimization, equilibria, etc. To illustrate it, let us consider a local optimal solution $\bar{x}$ to the following problem of *constrained optimization*:

$$\text{minimize } \ \varphi(x) \ \text{ subject to } \ x \in \Omega \subset X.$$

Then one can easily check that $(\bar{x}, \varphi(\bar{x}))$ is a local extremal point of the set system $\{\Omega_1, \Omega_2\}$ in $X \times I\!R$ with $\Omega_1 = \operatorname{epi}\varphi$ and $\Omega_2 = \Omega \times \{\varphi(\bar{x})\}$. More examples of extremal systems of sets related and also not related to optimization can be found in [12,13],

It is not hard to observe that $\bar{x} \in \Omega_1 \cap \Omega_2$ is a *local extremal point* of the set system $\{\Omega_1, \Omega_2\}$ *if and only if*

$$\vartheta\big(\Omega_1 \cap B_r(\bar{x}), \Omega_2 \cap B_r(\bar{x})\big) = 0 \quad \text{for some} \quad r > 0, \tag{12}$$

where the *measure of overlapping* $\vartheta(\Omega_1, \Omega_2)$ for the sets Ω_1, Ω_2 is defined by

$$\vartheta(\Omega_1, \Omega_2) := \sup\big\{\nu \geq 0 \big|\ \nu I\!B \subset \Omega_1 - \Omega_2\big\}$$

Modifying the constant $\vartheta(\cdot, \cdot)$ in (12), Kruger introduced (under the name of "extended extremality" in [7] and "weak stationarity" in [8]) the new notion of extremality for set systems that in fact reflects a certain amount of *linear subextremality*; see Sect. 1 and the discussion below.

Definition 1 *(linear subextremality of sets)* Given $\Omega_1, \Omega_2 \subset X$ and $\bar{x} \in \Omega_1 \cap \Omega_2$, we say that the set system $\{\Omega_1, \Omega_2\}$ is LINEARLY SUBEXTREMAL around the point $\bar{x}$ if

$$\vartheta_{\text{lin}}(\Omega_1, \Omega_2, \bar{x}) := \liminf_{\substack{x_i \xrightarrow{\Omega_i} \bar{x} \\ r \downarrow 0}} \frac{\vartheta\big([\Omega_1 - x_1] \cap r I\!B, [\Omega_2 - x_2] \cap r I\!B\big)}{r} = 0 \tag{13}$$

with $i = 1, 2$ under the lim inf sign in (13).

It is clear that the set extremality in the sense of (12) implies the linear subextremality in the sense of (13), but not vice versa. Let us discuss some specific features of linear subextremality for set systems that distinguish this notion from the concept of (12).

(a) The constant $\vartheta_{\text{lin}}(\Omega_1, \Omega_2, \bar{x})$ defined in (13), in contrast to the one from (12), involves a *linear rate* of set perturbations as $r \downarrow 0$. Therefore, condition (13) describes a *local non-overlapping at linear rate* for the sets Ω_1 and Ω_2, while condition (12) corresponds to a local non-overlapping of these sets with an *arbitrary rate* as $r \downarrow 0$,

(b) Condition (13) requires not the *precise* local non-overlapping of the given sets but up to their *infinitesimally small deformations*.
(c) Condition (13) does not require that the sets Ω_1 and Ω_2 non-overlap *exactly at* the point $\bar{x}$. Moreover, it is easy to observe from the relations in (b) that (13) holds if, given any neighborhood U of $\bar{x}$, there are points $x_1 \in \Omega_1 \cap U$ and $x_2 \in \Omega_2 \cap U$ ensuring an *approximate non-overlapping* of the translated sets $\Omega_1 - x_1$ and $\Omega_2 - x_2$ with a linear rate.

One of the most important results in the geometric theory of variational analysis and its applications is the so-called *extremal principle* providing *necessary* conditions for local extremal points of closed set systems. Its first versions were formulated in [9], while the most advanced result on the *exact* (pointbased) extremal principle is given in [12, Theorem 2.22]: *Let $\bar{x} \in \Omega_1 \cap \Omega_2$ be a local extremal point of the set system $\{\Omega_1, \Omega_2\}$, where Ω_1 and Ω_2 are locally closed subsets of an Asplund space X. Assume that either Ω_1 or Ω_2 is SNC at $\bar{x}$. Then there is $x^* \in X^*$ satisfying*

$$x^* \in N(\bar{x}; \Omega_1) \cap (-N(\bar{x}; \Omega_2)), \quad \|x^*\| = 1. \tag{14}$$

The following theorem shows that the above conditions of the extremal principle are *necessary* not only for local extremal points of $\{\Omega_1, \Omega_2\}$ but also for a *weaker* (less restrictive) notion of *linear subextremality*, providing actually a *characterization* of Asplund spaces. Moreover, these conditions happen to be *necessary and sufficient* for linear subextremality in finite-dimensional spaces.

Theorem 1 (linear subextremality via the extremal principle) *Let Ω_1 and Ω_2 be nonempty subsets of a Banach space X, and let $\bar{x} \in \Omega_1 \cap \Omega_2$. Assume that both Ω_1 and Ω_2 are locally closed around $\bar{x}$ and that one of them is sequentially normally compact at this point. The following assertions hold:*

(i) *If X is Asplund and if the system $\{\Omega_1, \Omega_2\}$ is linearly subextremal around $\bar{x}$, then there is $x^* \in X^*$ satisfying the relationships of the extremal principle* (14).
(ii) *Furthermore, if relationships* (14) *are satisfied for every set system $\{\Omega_1, \Omega_2\}$ linearly subextremal around $\bar{x}$, then the space X is Asplund.*
(iii) *Let* $\dim X < \infty$. *Then the system $\{\Omega_1, \Omega_2\}$ is linearly subextremal around $\bar{x}$ if and only if the relationships of the extremal principle* (14) *are satisfied.*

Proof To justify (i), consider arbitrary subsets Ω_1 and Ω_2 of an Asplund space X that are locally closed around $\bar{x}$ and form a linearly subextremal system around this point. Let $\varepsilon > 0$. As observed in [7], a slight modification of the proof of the corresponding (approximate) necessary conditions for local extremal points (12) leads us in fact to the following result concerning linear subextremality (cf. [12, Subsect. 2.2.3] and [13, Subsect. 5.4.1] for more details and discussions): there are $x_i \in \Omega_i \cap (\bar{x} + \varepsilon I\!B)$ and $x_i^* \in \widehat{N}(x_i; \Omega_i)$, $i = 1, 2$, satisfying the relationships

$$\|x_1^* + x_2^*\| \le \varepsilon \;\text{ and }\; \|x_1^*\| + \|x_2^*\| = 1.$$

Now picking $\varepsilon_k \downarrow 0$ as $k \to \infty$ and using the latter result, we find sequences $x_{ik} \to \bar{x}$ with $x_{ik} \in \Omega$ and $x_{ik}^* \in \widehat{N}(x_{ik}; \Omega_i)$ as $i = 1, 2$ such that

$$\|x_{1k}^* + x_{2k}^*\| \le \varepsilon_k \;\text{ and }\; \|x_{1k}^*\| + \|x_{2k}^*\| = 1 \;\text{ whenever }\; k \in I\!N. \tag{15}$$

Since bounded subsets in duals to Asplund spaces are weak* sequentially compact [5], there are subsequences of $\{x_{1k}^*\}$ and $\{x_{2k}^*\}$, which weak* converge to some $x_1^* \in X^*$ and $x_2^* \in X^*$, respectively. Passing to the limit as $k \to \infty$ in the first relationship of (15) and taking into account the well-known lower semicontinuity of the normal function $\|\cdot\|$ in the weak* topology of X^*, we conclude that $x_1^* = -x_2^* =: x^*$. Furthermore,

$$x^* \in N(\bar{x}; \Omega_1) \cap (-N(\bar{x}; \Omega_2))$$

by definition (2) of the basic normal cone. To prove assertion (i) of the theorem, it remains to show that $x^* \neq 0$ if one of the sets Ω_i (say Ω_1 for definiteness) is SNC at $\bar{x}$.

By the contrary, assume that $x^* = 0$. Then $x_{1k}^* \xrightarrow{w^*} 0$ and hence $\|x_{1k}^*\| \to 0$ as $k \to \infty$ by the SNC property of Ω_1 at $\bar{x}$. Employing the first relationship in (15), we get that $\|x_{2k}^*\| \to 0$ as well. This obviously contradicts the second (non-triviality) relationship in (15) for large $k \in I\!N$ and thus completes the proof of assertion (i) of the theorem.

Assertion (ii) follows directly from Theorem 2.22(ii) in [12], which justifies the Asplund property of X provided that (14) holds for every local extremal point $\bar{x}$ of an arbitrary set system $\{\Omega_1, \Omega_2\}$ that are locally closed around $\bar{x}$ and one of which is SNC at $\bar{x}$. As observed above, the local extremality of $\{\Omega_1, \Omega_2\}$ at $\bar{x}$ in the sense of (12) implies the linear subextremality of $\{\Omega_1, \Omega_2\}$ around this point in the sense of Definition 1.

Let us finally justify assertion (iii). Due to the above conclusion of (i) and the automatic fulfillment of the SNC property in finite dimensions, it remains to show that the relationships (14) of the (exact) extremal principle imply the linear subextremality of $\{\Omega_1, \Omega_2\}$ around $\bar{x}$ provided that $\dim X < \infty$. To proceed, we take $\bar{x} \in \Omega_1 \cap \Omega_2$ satisfying (14) and, using the construction (2) in finite dimensions, find sequences $x_{ik} \to \bar{x}$, $x_{1k}^* \to x^*$, and $x_{2k}^* \to -x^*$ as $k \to \infty$ such that

$$x_{ik} \in \Omega_i \;\text{ and }\; x_{ik}^* \in \widehat{N}(x_{ik}; \Omega_i) \quad \text{for } i = 1, 2,\; k \in I\!N.$$

Since $\|x_{1k}^*\| + \|x_{2k}^*\| \to 2\|x^*\| = 2$ as $k \to \infty$, we get by the standard normalization procedure that for every $\varepsilon > 0$ there are

$$\begin{aligned}&x_i \in \Omega_i \cap (\bar{x} + \varepsilon I\!B) \;\text{ and }\; x_i^* \in \widehat{N}(x_i; \Omega_i) \;\text{ as }\; i = 1, 2\\ &\text{with }\; \|x_1^* + x_2^*\| \le \varepsilon \;\text{ and }\; \|x_1^*\| + \|x_2^*\| = 1.\end{aligned}$$

Employing now [7, Theorem 4.1], we conclude that the set system $\{\Omega_1, \Omega_2\}$ is linearly subextremal around $\bar{x}$ and thus complete the proof of the theorem. □

Note that the above proof of assertion (iii) of Theorem 1 in the general geometric setting essentially employs the *finite dimensionality* of the space X ensuring that the weak* topology of X^* agrees with the norm one. By the fundamental Josefson-Nissenzweig theorem, this never holds in a Banach space of infinite dimension. Nevertheless, the latter finite dimensionality assumption can be essentially relaxed (partially dropped) for sets Ω_i of certain special functional structures considered in the next sections.

4 Linear suboptimality in multiobjective problems and EPECs

The main attention in this section is paid to the notion of *linearly suboptimal solutions* to problems of *constrained multiobjective optimization*. This non-conventional notion of suboptimality is actually induced, in the functional framework, by the geometric concept of linear subextremality for set systems studied in Sect. 3.

Given a mapping $f: X \to Z$ between Banach spaces, subsets $\Omega \subset X$ and $\Theta \subset Z$, and a point $\bar{x} \in \Omega$, we consider following [7] the constant

$$\vartheta_{\text{lin}}(f, \Omega, \Theta, \bar{x}) := \liminf_{\substack{x \xrightarrow{\Omega} \bar{x},\, z \xrightarrow{\Theta} f(x) \\ r \downarrow 0}} \frac{\vartheta\big(f(B_r(x) \cap \Omega) - f(x), \Theta - z\big)}{r}, \tag{16}$$

and introduce the notion of *linear suboptimality* as in [13], which was originally defined as "(f, Ω, Θ)-extremality" in [7] and as "weak stationarity" in [8].

Definition 2 (*linearly suboptimal solutions to multiobjective problems*) Given $(f, \Omega, \Theta, \bar{x})$ as above, we say that the point $\bar{x} \in \Omega$ is LINEARLY SUBOPTIMAL with respect to (f, Ω, Θ) if one has

$$\vartheta_{\text{lin}}(f, \Omega, \Theta, \bar{x}) = 0.$$

It is easy to check that the point $\bar{x} \in \Omega$ is linearly suboptimal in the sense of Definition 2 if it is (locally) (f, Ω, Θ)-*optimal* in the following sense: there is a neighborhood U of $\bar{x}$ and a sequence $\{z_k\} \subset Z$ with $\|z_k\| \to 0$ as $k \to \infty$ such that

$$f(x) - f(\bar{x}) \notin \Theta - z_k \quad \text{for all} \quad x \in \Omega \cap U \quad \text{and} \quad k \in I\!N,$$

where we may always put $f(\bar{x}) = 0$ for simplicity and assume that $0 \in \Theta$. The latter notion is induced by the concept of local extremal points for set systems, discussed in the beginning of Sect. 3, and cover various concepts of optimality in multiobjective optimization; see [13, Subsect. 5.3.1] for more details, examples, and discussions.

In what follows, we derive pointwise/pointbased *necessary and sufficient conditions* for linear (f, Ω, Θ)-suboptimality expressed in terms of our basic/limiting normal and coderivative constructions defined in Sect. 2.

Let us start with the so-called "condensed" conditions expressed via coderivatives of the set-valued mapping

$$F(x) := \begin{cases} f(x) - \Theta & \text{if } x \in \Omega, \\ \emptyset & \text{otherwise.} \end{cases} \tag{17}$$

built upon the initial data of (16).

Theorem 2 (condensed necessary and sufficient conditions for linear suboptimality in multiobjective problems) *Let F be a mapping between Banach spaces defined in* (17). *The following assertions hold:*

(i) *Assume that* $\dim X < \infty$ *and that there is* $0 \neq z^* \in Z^*$ *satisfying*

$$0 \in D^*_M F(\bar{x}, 0)(z^*).$$

Then $\bar{x}$ is linearly suboptimal with respect to (f, Ω, Θ).

(ii) *Conversely, assume that $\bar{x}$ is linearly suboptimal with respect to (f, Ω, Θ). Then there is $0 \neq z^* \in Z^*$ satisfying*

$$0 \in \widetilde{D}^*_M F(\bar{x}, 0)(z^*)$$

provided that both X and Z are Asplund, that $\operatorname{gph} F$ *is locally closed around $(\bar{x}, 0)$, and that F^{-1} is PSNC at $(0, \bar{x})$; the latter is automatic when* $\dim Z < \infty$.

(iii) *Let* $\dim X < \infty$, *let* Z *be Asplund, and let* F *be closed-graph around* $(\bar{x}, 0)$. *Assume also that* F *is SNC and strongly coderivatively normal at* $(\bar{x}, 0)$ *with*

$$D^*F(\bar{x}, 0) := D^*_M F(\bar{x}, 0) = D^*_N F(\bar{x}, 0).$$

Then $\bar{x}$ *is linearly suboptimal with respect to* (f, Ω, Θ) *if and only if there is* $0 \neq z^* \in Z^*$ *satisfying the inclusion*

$$0 \in D^*F(\bar{x}, 0)(z^*).$$

Proof Let us start with justifying assertion (i). Using $0 \in D^*_M F(\bar{x}, 0)(z^*)$ and the definition of the mixed coderivative (6) in the case of $\dim X < \infty$, we find $\varepsilon_k \downarrow 0$, $x_k \to \bar{x}$, $z_k \to 0$, $x_k^* \to 0$, and $z_k^* \to z^*$ such that

$$z_k \in F(x_k) \quad \text{and} \quad x_k^* \in \widehat{D}^*_{\varepsilon_k} F(x_k, z_k)(z_k^*) \quad \text{whenever} \quad k \in I\!N.$$

Note that the first inclusion above implies, by the construction of F in (17), that $x_k \in \Omega$ and $z_k = f(x_k) \in \Theta$. Furthermore, since $\|z_k^* - z^*\| \to 0$ as $k \to \infty$ and $\|z^*\| = 1$, we can assume without loss of generality that $\|z_k^*\| = 1$ for each $k \in I\!N$. From the relation $x_k^* \in \widehat{D}^*_{\varepsilon_k} F(x_k, z_k)(z_k^*)$ one has

$$\langle x_k^*, x - x_k\rangle - \langle z_k^*, z - z_k\rangle \leq \varepsilon_k\big(\|x - x_k\| + \|z - z_k\|\big)$$

whenever the pair (x, z) is sufficiently close to $(\bar{x}, 0)$. This implies the estimate

$$-\langle z_k^*, z - z_k\rangle \leq \big(\varepsilon_k + \|x_k^*\|\big)\big(\|x - x_k\| + \|z - z_k\|\big),$$

which means that

$$0 \in \widehat{D}^*_{\gamma_k} F(x_k, z_k)(z_k^*) \quad \text{with} \quad \gamma_k := \varepsilon_k + \|x_k^*\| \downarrow 0 \quad \text{as} \quad k \to \infty. \tag{18}$$

It is easy to see that the point $\bar{x}$ is linearly suboptimal with respect to (f, Ω, Θ) *if and only if* the system of two sets

$$\Omega_1 := \text{gph}\, F \quad \text{and} \quad \Omega_2 := X \times \{0\} \subset X \times Z$$

is linearly subextremal around $(\bar{x}, 0) \in X \times Z$ in the sense of Definition 1. Observe that

$$\widehat{N}_\varepsilon((x, 0); \Omega_2) = (\varepsilon I\!B^*) \times Z^*, \quad \varepsilon > 0,$$

and that the inclusion in (18) is equivalent to

$$(0, -z_k^*) \in \widehat{N}_{\gamma_k}((x_k, z_k); \Omega_1) \ \text{ for all } \ k \in I\!N.$$

Then the proof of the sufficiency part of [7, Theorem 4.1] ensures that, in the general Banach space framework, $\{\Omega_1, \Omega_2\}$ is linearly subextremal around $(\bar{x}, 0)$, and thus $\bar{x}$ is linearly suboptimal with respect to (f, Ω, Θ).

To prove assertion (ii), we fix a point $\bar{x}$ linearly suboptimal with respect to (f, Ω, Θ) and pick an arbitrary sequence $\varepsilon_k \downarrow 0$ as $k \to \infty$. Taking into account the above discussions and the proof of assertions (i) of Theorem 1, find sequences $(x_k, z_k) \to (\bar{x}, 0)$ with $z_k \in F(x_k)$ and $z_k^* \in Z^*$ with $\|z_k^*\| = 1$ satisfying $0 \in \widehat{D}^* F(x_k, z_k)(z_k^*)$ for all $k \in I\!N$. Since Z is Asplund, there is $z^* \in Z^*$ such that $z_k^* \xrightarrow{w^*} z^*$ as $k \to \infty$ along a subsequence, and one clearly has $0 \in \widetilde{D}^*_M F(\bar{x}, 0)(z^*)$ by passing to the limit as $k \to \infty$. Furthermore, $z^* \neq 0$ by the PSNC assumption imposed. The latter assumption obviously holds if Z is finite-dimensional. Thus we arrive at the conclusion of (ii).

The final assertion (iii) is a direct combination of (i) and (ii) by the assumptions made, which are discussed in Sect. 2. Note that $\widetilde{D}^*_M F(\bar{x}, 0) = D^*_N F(\bar{x}, 0)$ and the PSNC property of F^{-1} is equivalent to the SNC property of F in this case, since $\dim X$ is finite-dimensional. This completes the proof of the theorem. □

Using extended calculus rules available for the generalized differentiable constructions and SNC/PSNC properties involved in the formulation of Theorem 2 (see [12]), we can deduce from the condensed results in assertion (ii) of this theorem comprehensive *necessary* conditions for *linear suboptimality* in multiobjective problems and their specifications subject to various (in particular, equilibrium) constraints expressed *separately* via the initial data (f, Ω, Θ), i.e., in terms of generalized differential constructions for each of f, Ω, and Θ; cf. the results of Subsects. 5.3.2 and 5.3.5 in [13] for the case of necessary conditions for (f, Ω, Θ)-*optimality*. The situation for *sufficient* conditions and also for the *characterization* of linear suboptimality is more delicate: we have to employ calculus rules with *equalities*, which are essentially more restrictive than those we need for necessity. Let us present some results in this direction providing the characterization of linear suboptimality in terms of the initial data (f, Ω, Θ) based on the condensed conditions of Theorem 2(iii).

Recall that $f: X \to Y$ is *strictly Lipschitzian* at $\bar{x}$ if it is locally Lipschitzian around this point and there is a neighborhood V of the origin in X such that the sequence

$$y_k := \frac{f(x_k + t_k v) - f(x_k)}{t_k}, \quad k \in I\!N,$$

contains a norm convergent subsequence whenever $v \in V$, $x_k \to \bar{x}$, and $t_k \downarrow 0$. It obviously reduces to the standard local Lipschitzian property of f around $\bar{x}$ when Y is finite-dimensional; see [12, Subsect. 5.1.3] for a detailed study and applications of this property in infinite-dimensional spaces.

Theorem 3 (separated criteria for linear suboptimality in multiobjective problems) *Let $f: X \to Z$ be Lipschitz continuous around $\bar{x}$ with* $\dim X < \infty$*, and let $\Omega \subset X$ and $\Theta \subset Z$ be locally closed around $\bar{x}$ and $\bar{z} := f(\bar{x}) \in \Theta$, respectively. Impose one of the following assumptions* (a)–(c) *on the initial data:*

(a) $\dim Z < \infty$ *and either $\Omega = X$, or f strictly differentiable at $\bar{x}$.*
(b) *Z is Asplund, $\Omega = X$, Θ is normally regular and SNC at $\bar{z}$, and f is strictly Lipschitzian at $\bar{x}$.*
(c) *Z is Asplund, Ω is normally regular at $\bar{x}$, Θ is normally regular and SNC at $\bar{z}$, and f is N-regular at $\bar{x}$.*

Then $\bar{x}$ is linearly suboptimal with respect to (f, Ω, Θ) if and only if there is $0 \neq z^* \in Z^*$ satisfying the conditions

$$0 \in \partial\langle z^*, f\rangle(\bar{x}) + N(\bar{x}; \Omega), \quad z^* \in N(\bar{z}; \Theta).$$

Proof Given (f, Ω, Θ) in the theorem, consider the set

$$\mathcal{E}(f, \Omega, \Theta) := \{(x, z) \in X \times Z \big|\ f(x) - z \in \Theta,\ x \in \Omega\big\}$$

and observe that this set is the graph of the mapping F defined in (17). Hence

$$D^*_N F(\bar{x}, 0)(z^*) = \big\{x^* \in X^* \big|\ (x^*, -z^*) \in N\big((\bar{x}, 0); \mathcal{E}(f, \Omega, \Theta)\big)\big\},$$

where $f_\Omega(x) := f(x)+\delta(x;\Omega)$ stands for the *restriction* of f on the set Ω. Then Lemma 5.23 from [13] ensures the representation

$$D_N^* F(\bar{x},0)(z^*) = \begin{cases} \partial\langle z^*, f_\Omega\rangle(\bar{x}) & \text{if } z^* \in N(\bar{z};\Theta), \\ \emptyset & \text{otherwise} \end{cases} \tag{19}$$

provided that Z is Asplund and that f_Ω is locally Lipschitzian around $\bar{x}$ and strongly coderivatively normal at this point.

Let us first justify in parallel assertions (a) and (b) of the theorem, where $\Omega = X$. It follows from (19) and [12, Proposition 4.9] that F is strongly coderivatively normal at $(\bar{x}, 0)$ if either $\dim Z < \infty$, or f is strictly Lipschitzian at $\bar{x}$ and Θ is normally regular at $\bar{z}$. To meet all the assumptions of Theorem 2 stated above, we need also checking (in the case of $\dim Z = \infty$) that F^{-1} is PSNC at $(0, \bar{x})$. Theorem 5.59 from [13] ensures this property if either Θ is SNC at $\bar{z}$ or f^{-1} is PSNC at $(\bar{z}, \bar{x})$. Since X is finite-dimensional, the latter is equivalent to the SNC property of f at $(\bar{x}, \bar{z})$ and, by [12, Corollary 3.30], it reduces to $\dim Z < \infty$ for strictly Lipschitzian mappings. Thus we complete the proof of the theorem in the case of $\Omega = X$.

To proceed in the constraint case of $\Omega \neq X$ in assertion (c), it remains to justify that

$$\partial\langle z^*, f_\Omega\rangle(\bar{x}) = \partial\langle z^*, f\rangle(\bar{x}) + N(\bar{x};\Omega) \tag{20}$$

in (19) under the assumptions made in (c). If f is strictly differentiable at $\bar{x}$, (20) follows directly from the easy subdifferential sum rule held in this case. The more involved sum rule from [12, Proposition 3.12] ensures (20) and also the N-regularity (and hence the coderivative normality) of the restriction f_Ω at $\bar{x}$ when f is N-regular and Ω is normally regular at this point. Combining these facts with the assumptions on Θ in (c) needed in the case of $\dim Z = \infty$ similarly to the above proof for $\Omega = X$, we arrive at all the requirements of Theorem 2(iii) and thus complete the proof of the theorem. $\square$

Let us present two corollaries of Theorem 3 in the case of multiobjective problems with constraint sets given in more specific forms typical in applications. First consider the case of the so-called *operator constraints* defined by

$$\Omega = g^{-1}(\Lambda) \ \text{ with } \ g\colon X \to Y \ \text{ and } \ \Lambda \subset Y. \tag{21}$$

Corollary 1 (pointbased criteria for linear suboptimality under operator constraints) *Let in the framework of Theorem 3 the constraint set Ω be given by (21). Assume that* $\dim X < \infty$, *that Θ and Λ are locally closed around $\bar{z}$ and $\bar{y} := g(\bar{x})$, respectively, and that f and g are strictly differentiable at $\bar{x}$. Suppose also that one of the following assumptions holds:*

(a) *Y is Banach,* $\dim Z < \infty$, *and $\nabla g(\bar{x})$ is surjective.*
(b) $\dim Y < \infty$, *Z is Asplund, Λ is normally regular at $\bar{y}$, Θ is normally regular and SNC at $\bar{z}$, and*

$$N(\bar{y};\Lambda) \cap \ker \nabla g(\bar{x})^* = \{0\}.$$

Then $\bar{x}$ is linearly suboptimal with respect to $(f, g^{-1}(\Lambda), \Theta)$ if and only if there is $0 \neq z^* \in Z^*$ satisfying the relations

$$0 \in \nabla f(\bar{x})^* z^* + \nabla g(\bar{x})^* N(\bar{y};\Lambda), \quad z^* \in N(\bar{z};\Theta).$$

Proof We use Theorem 3 with $\Omega := g^{-1}(\Lambda)$. First apply [12, Theorem 1.17] to ensure the calculus formula

$$N(\bar{x};\Omega) = \nabla g(\bar{x})^* N(\bar{y};\Lambda)$$

under the surjectivity assumption on $\nabla g(\bar{x})$ made in (a) when Y is Banach. Then we arrive at the conclusion of this corollary due to Theorem 3(a).

To ensure the normal regularity of $\Omega = g^{-1}(\Lambda)$, needed in Theorem 3(c) in addition to the above calculus formula, we employ [12, Theorem 3.13(iii)] with $F(y) = \delta(y; \Lambda)$ therein, which justifies the conclusion of the corollary under the assumptions made in (b). Note that we cannot get anything but strict differentiability from the N-regularity condition on g in the latter theorem, since the graphical regularity of g is equivalent to its strict differentiability at the reference point due to [12, Corollary 3.69] with $\dim X < \infty$. □

The result obtained has a striking consequence for the case of multiobjective problems with *functional constraints* in the classical form of equalities and inequalities given by strictly differentiable functions. In this case an appropriate multiobjective version of the *Lagrange multiplier rule* in *normal form* provides *necessary and sufficient* conditions for linear suboptimality under the *Mangasarian-Fromovitz constraint qualification*.

Corollary 2 (linear suboptimality in multiobjective problems with functional constraints) *Let $f: X \to Z$ be strictly differentiable at $\bar{x}$ with $\dim X < \infty$ and Z being Asplund, let Θ be normally regular and SNC at $\bar{z}$, and let*

$$\Omega := \left\{x \in X \,\middle|\, \varphi_i(x) \le 0,\ i = 1, \dots, m;\ \varphi_i(x) = 0,\ i = m+1, \dots, m+r\right\},$$

where each φ_i is strictly differentiable at $\bar{x}$. Assume the Mangasarian-Fromovitz constraint qualification:

(a) $\nabla\varphi_{m+1}(\bar{x}), \dots, \nabla\varphi_{m+r}(\bar{x})$ *are linearly independent, and*
(b) *there is $u \in X$ satisfying*

$$\langle\nabla\varphi_i(\bar{x}), u\rangle < 0, \quad i \in \{1, \dots, m\} \cap I(\bar{x}),$$

$$\langle\nabla\varphi_i(\bar{x}), u\rangle = 0, \quad i = m+1, \dots, m+r,$$

where $I(\bar{x}) := \left\{i = 1, \dots, m+r \,\middle|\, \varphi_i(\bar{x}) = 0\right\}$.

Then $\bar{x}$ is linearly suboptimal with respect to (f, Ω, Θ) if and only if there is $z^* \in N(\bar{z}; \Theta)\setminus\{0\}$ and $(\lambda_1, \dots, \lambda_{m+r}) \in I\!R^{m+r}$ such that

$$\nabla f(\bar{x})^* z^* + \sum_{i=1}^{m+r} \lambda_i \nabla\varphi_i(\bar{x}) = 0,$$

$$\lambda_i \ge 0 \ \ \text{and} \ \ \lambda_i\varphi_i(\bar{x}) = 0 \ \ \text{for all} \ \ i = 1, \dots, m.$$

Proof Follows from Corollary 1(b) with

$$\Lambda := \left\{(\alpha_1, \dots, \alpha_{m+r}) \in I\!R^{m+r} \,\middle|\, \begin{array}{l} \alpha_i \le 0 \ \text{ for } \ i = 1, \dots, m \quad \text{and} \\[2ex] \alpha_i = 0 \ \text{ for } \ i = m+1, \ \dots, m+r \end{array}\right\}$$

and $g := (\varphi_1, \ \dots, \varphi_{m+r}): X \to I\!R^{m+r}$. □

Finally in this section, we consider a rather new class of optimization-related problems known as *equilibrium problems with equilibrium constraints* (EPECs). In [13,14] and the references therein, the reader can find more information about such problems, their modifications, and interpretations from the viewpoint of multiobjective optimization. Our main goal is to derive *necessary and sufficient* conditions for *linear suboptimality* in EPECs described as follows:

Given $f: X \times Y \to Z$, $S: X \rightrightarrows Y$, and $\Theta \subset Z$, we say that $(\bar{x}, \bar{y})$ is *linearly suboptimal* with respect to (f, S, Θ) if it is linearly suboptimal with respect to $(f, \operatorname{gph} S, \Theta)$ in the sense of Definition 2. We are mostly interested in equilibrium constraints given by solution maps to *parametric variational systems* of the type

$$S(x) := \big\{ y \in Y \big|\, 0 \in q(x, y) + Q(x, y) \big\}. \tag{22}$$

where $q: X \times Y \to P$ is single-valued while $Q: X \times Y \rightrightarrows P$ is a set-valued mapping between Banach spaces. For $Q = Q(y)$, model (22) corresponds to the so-called *generalized equations* in the sense of Robinson [16], which reduce to the classical *variational inequalities* when $Q(y) = N(y; \Omega)$ is the normal cone mapping generated by a convex set Ω.

First observe, based on Theorem 2(ii) and calculus rules of the inclusion type, that all the *necessary* conditions obtained in [13, Subsect. 5.3.5] for generalized order optimality hold true for linearly suboptimal solutions to the EPECs under consideration. To derive *criteria* for linear suboptimality, we need to employ more restrictive calculus rules of the *equality* type that provide exact formulas for computing coderivatives of solution maps given by equilibrium constraints and also ensure *graphical regularity* of these maps in appropriate settings. To proceed, we rely on the results of Theorem 3(c) with $\Omega = \operatorname{gph} S \subset X \times Y$ and on the corresponding coderivative formulas and regularity assertions established in [12, Subsect. 4.4.1] for parametric variational systems. In the next theorem we impose for the strict differentiability assumption on f, which is actually not far from its N-regularity imposed in Theorem 4.3(c); these properties always agree when Z is finite-dimensional.

Theorem 4 (characterization of linear suboptimality in general EPECs) *Let $f: \mathbb{R}^n \times \mathbb{R}^m \to Z$ and $q: \mathbb{R}^n \times \mathbb{R}^m \to P$ be strictly differentiable at $(\bar{x}, \bar{y})$ with $\bar{z} := f(\bar{x}, \bar{y}) \in \Theta$ and $\bar{p} := -q(\bar{x}, \bar{y})$; let $\Theta \subset Z$ and the graph of $Q: \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows P$ be locally closed around $\bar{z}$ and $(\bar{x}, \bar{y}, \bar{p}) \in \operatorname{gph} Q$, respectively; and let $S: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be given in (22). Assume in addition that one of the following requirements holds:*

(a) $\dim Z < \infty$, *P is Banach, $\nabla_x q(\bar{x}, \bar{y})$ is surjective, and $Q = Q(y)$;*
(b) *Z and P are Asplund, Θ is SNC and normally regular at $\bar{z}$, $Q = Q(x, y)$ is SNC and N-regular at $(\bar{x}, \bar{y}, \bar{p})$, and the adjoint generalized equation*

$$0 \in \nabla q(\bar{x}, \bar{y})^* p^* + D_N^* Q(\bar{x}, \bar{y}, \bar{p})(p^*) \tag{23}$$

has only the trivial solution $p^* = 0$.

Then $(\bar{x}, \bar{y})$ is linearly suboptimal with respect to (f, S, Θ) if and only if there are linear functionals $z^* \in N(\bar{z}; \Theta) \setminus \{0\}$ and $p^* \in P^*$ satisfying

$$0 \in \nabla f(\bar{x}, \bar{y})^* z^* + \nabla q(\bar{x}, \bar{y})^* p^* + D_N^* Q(\bar{x}, \bar{y}, \bar{p})(p^*).$$

Proof Employing Theorem 3 with $\Omega = \operatorname{gph} S \subset X \times Y$, we conclude that $(\bar{x}, \bar{y})$ is linearly suboptimal with respect to (f, S, Θ) *if and only if* there is $z^* \in N(\bar{z}; \Theta) \setminus \{0\}$ satisfying

$$0 \in \nabla f(\bar{x}, \bar{y})^* z^* + N\big((\bar{x}, \bar{y}); \operatorname{gph} S\big)$$

provided that both X and Y are finite-dimensional, that f is strictly differentiable at $(\bar{x}, \bar{y})$, and that either $\dim Z < \infty$ or Z is Asplund, Θ is SNC and normally regular at $\bar{z}$, and S is N-regular at $(\bar{x}, \bar{y})$.

To obtain results in terms of the initial data for the solution map S, we need to represent $N\big((\bar{x},\bar{y});\operatorname{gph} S\big)$ via (q,Q) and also to invoke additional conditions ensuring the N-regularity of S at $(\bar{x},\bar{y})$ when $\dim Z=\infty$. First consider the case of $\dim Z<\infty$, when we do not need to ensure the regularity of S. In this case one has by [12, Theorem 4.44(i)] that

$$N\big((\bar{x},\bar{y});\operatorname{gph} S\big)=\Big\{(x^*,y^*)\in X^*\times Y^*\Big|\ x^*=\nabla_x q(\bar{x},\bar{y})^*p^*,\\ y^*\in\nabla_y q(\bar{x},\bar{y})^*p^*+D_N^*Q(\bar{y},\bar{p})(p^*)\ \text{ for some }\ p^*\in P^*\Big\}$$

when P is Banach, $Q=Q(y)$, and $\nabla_x q(\bar{x},\bar{y})$ is surjective. This gives the conclusion of the theorem in case (a).

If $Q=Q(x,y)$ and Z is Asplund, we employ [12, Theorem 4.44(ii)], which gives the representation formula for $N\big((\bar{x},\bar{y});\operatorname{gph} S\big)$ and simultaneously ensures the N-regularity of solution map S at $(\bar{x},\bar{y})$ under the N-regularity assumption on mapping Q at $(\bar{x},\bar{y},\bar{p})$ but with no surjectivity of the partial derivative $\nabla_x q(\bar{x},\bar{y})$. Combining this with the assumptions in Theorem 3(c), we complete the proof of the theorem. □

The most restrictive assumption in Theorem 4(b) is the N-regularity of Q at the reference point. It particularly holds when Q is *convex-graph*, in which case the conditions of Theorem 4 can be expressed *explicitly* in terms of Q instead of its coderivative; cf. [12, Corollary 4.45]. Other specifications of Theorem 4 can be derived for EPECs whose equilibrium constraints of (22) are described in the *composite subdifferential forms*

$$Q(x,y)=\partial(\psi\circ g)(x,y)\ \text{ or }\ Q(x,y)=(\partial\psi\circ g)(x,y) \tag{24}$$

typical in applications, where $g\colon I\!\!R^n\times I\!\!R^m\to W$ and $\psi\colon W\to\overline{I\!\!R}$ in the framework of Theorem 4 with a Banach space W. The results obtained in both settings of (24) are based on the *second-order subdifferential calculus* developed in [12]; cf. also the next section for some counterparts in the case of mathematical programs with equilibrium constraints.

5 Linear subminimality with applications to MPECs

In the concluding section of this paper, we study the notion of linear suboptimality from Definition 2 in the particular case of usual *minimization* problems; thus we refer to this notion as to *linear subminimality*. Minimization problems form a special subclass of the multiobjective optimization problems of Sect. 4 with a single (real-valued) objective f and with $\Theta=I\!\!R_-$. On the other hand, such problems and their linearly suboptimal solutions have certain specific features in comparison with general problems of multiobjective optimization. In what follows, we derive *characterizing* results for linear subminimality in *pointbased* form for unconstrained and constrained minimization problems. A special attention is paid to applications of general subminimality results to the case of minimization problems with equilibrium constraints, i.e., for *MPECs*.

Definition 3 (*linear subminimality*) Let $\Omega\subset X$, and let $\varphi\colon X\to\overline{I\!\!R}$ be finite at $\bar{x}\in\Omega$. We say that $\bar{x}$ is LINEARLY SUBMINIMAL with respect to (φ,Ω) if

$$\limsup_{\substack{x\xrightarrow{\Omega}\bar{x}\\ \varphi(x)\to\varphi(\bar{x})\\ r\downarrow 0}}\ \inf_{u\in B_r(x)\cap\Omega}\frac{\varphi(u)-\varphi(x)}{r}=0.$$

The point $\bar{x}$ is said to be linearly subminimal for φ if $\Omega=X$ in the above.

This notion of linear subminimality corresponds to "almost minimality" in [7] and to "weak inf-stationarity" in [8], where a "fuzzy" subdifferential characterization of this property was derived in terms of Fréchet subgradients of the "condensed" function

$$\varphi_\Omega(x) := \varphi(x) + \delta(x; \Omega). \tag{25}$$

Our main goal is to obtain pointbased characterizations of linear subminimality in various problems of constrained minimization expressed via the initial data. This will be done by using extended *calculus rules* available for our basic/limiting normal and (first-order and second-order) subdifferential constructions.

The next theorem contains pointbased *necessary and sufficient* conditions for linear subminimality in general (non-structured) minimization problems with geometric constraints in finite-dimensional spaces.

Theorem 5 (pointbased criteria for linear subminimality) *Let* $\dim X < \infty$, *let* $\varphi\colon X \to \overline{I\!R}$ *be finite at* $\bar{x}$ *and l.s.c. around this point, and let* $\Omega \subset X$ *be locally closed around* $\bar{x}$. *The following assertions hold:*

(i) *The point* $\bar{x}$ *is linearly subminimal with respect to* (φ, Ω) *if and only if* $0 \in \partial\varphi_\Omega(\bar{x})$, *where* φ_Ω *is defined in* (25).

(ii) *Impose the alternative assumptions:*
   (a) *either* φ *is strictly differentiable at* $\bar{x}$,
   (b) *or* φ *is lower regular at* $\bar{x}$, Ω *is normally regular at* $\bar{x}$, *and*

$$\partial^\infty\varphi(\bar{x}) \cap \big(-N(\bar{x}; \Omega)\big) = \{0\}; \tag{26}$$

the latter qualification condition is automatics when φ *is locally Lipschitzian around* $\bar{x}$.

Then $\bar{x}$ *is linearly subminimal with respect to* (φ, Ω) *if and only if*

$$0 \in \partial\varphi(\bar{x}) + N(\bar{x}; \Omega). \tag{27}$$

Proof The condensed pointbased characterization in (i) follows from assertion (iii) of Theorem 4.2 with $\Theta = I\!R_-$, $f(x) = \varphi(x) - \varphi(\bar{x})$, and F defined in (17). Note that this F is automatically SNC and strongly coderivatively normal at $(\bar{x}, 0)$ due to $Z = I\!R$, and one obviously has the relationship

$$0 \in D^*F(\bar{x}, 0)(1) \Longleftrightarrow 0 \in \partial\varphi_\Omega(\bar{x}),$$

which reduces (i) to Theorem 4.2(iii).

To prove assertion (ii) of the theorem, we need to apply an *equality* sum rule for basic subgradients to the sum of functions in (25). When φ is strictly differentiable at $\bar{x}$, the required sum rule follows from [12, Proposition 1.107] with no regularity assumption on Ω. When (b) is assumed with the imposed regularity of φ and Ω, the sum rule

$$\partial\varphi_\Omega(\bar{x}) = \partial\varphi(\bar{x}) + N(\bar{x}; \Omega)$$

follows from [12, Theorem 3.36] under the qualification condition (26), which is automatic for locally Lipschitzian functions due to $\partial^\infty\varphi(\bar{x}) = \{0\}$ in this case; see [12, Corollary 1.81]. This justifies (27) and completes the proof of the theorem. □

Observe that in case (b) of Theorem 5.2(ii), the assumptions ensuring criterion (27) are essentially weaker than those induced by Theorem 4.3(c). Indeed, the N-regularity assumption on $f(x) = \varphi(x) - \varphi(\bar{x})$ with $Z = I\!R$ in Theorem 3(c), which is the graphical regularity of

φ at $\bar{x}$, is *equivalent* to the strict differentiability of φ at this point due to [12, Proposition 1.94]. On the other hand, the *lower regularity* of φ assumed in Theorem 5(b) holds for important classes of non-smooth functions encountered in minimization problems. In particular, this includes convex functions and a broader class of *amenable* functions widely discussed and applied in [17]. Such a difference between the results of Theorem 3 in the case of minimization problems and the ones of Theorem 5 is due to the *one-sided* specific character of minimizing extended-real-valued functions, which is missed by separated conditions in the vector/multiobjective framework.

Similarly to Sect. 4, we can derive from Theorem 5.2(ii) some consequences related to specific types of constraints. Note that the results obtained in this way are generally different from those established for constrained problems of multiobjective optimization; cf. Corollaries 4.4, 4.5 and Theorem 4.6, which unavoidably impose the strict differentiability assumption on the cost function.

Let us consider in more detail a broad class of *mathematical programs with equilibrium constraints* (MPECs) and its typical specifications important in various applications; see the books [10, 15] for comprehensive discussions.

General MPECs can be described in the following way:

$$\text{minimize } \ \varphi(x, y) \ \text{ subject to } \ y \in S(y), \quad (x, y) \in \Omega, \tag{28}$$

where $\varphi\colon X \times Y \to \overline{I\!R}$, $\Omega \subset X \times Y$, and $S\colon X \rightrightarrows Y$ is a set-valued mapping between Banach spaces, which is usually given in form (22) of solution maps to generalized equations/variational conditions. Similarly to Definition 5, one can formulate the notion of *linear subminimality* for such MPECs. Furthermore, based on the procedure employed above and generalized differential/SNC calculi, we observe that *all* the necessary conditions for conventional optimality for MPECs developed in [13, Sect. 5.2] *hold true as necessary conditions for linear suboptimality* in such problems.

In what follows, we focus on deriving *pointbased necessary and sufficient* conditions for linear suboptimality in MPECs (28) with the equilibrium constraints given by (22) putting $\Omega = X \times Y$ for simplicity. In this case we say that $(\bar{x}, \bar{y})$ is *MPEC linearly subminimal* in (28), (22) with respect to (φ, S) if it is linearly subminimal with respect to $(\varphi, \operatorname{gph} S)$ in the sense of Definition 5 considered in the product space $X \times Y$.

The next theorem is an MPEC counterpart of Theorem 4.6, where however we *significantly relax* the strict differentiability assumption on the cost function in case (b). Observe from the proof given below that the imposed Lipschitzian assumption on φ can be also relaxed by using an appropriate MPEC version of the qualification condition (26).

Theorem 6 (characterization of linear subminimality for general MPECs) *Let $\varphi\colon I\!R^n \times I\!R^m \to I\!R$ be locally Lipschitzian around $(\bar{x}, \bar{y})$, let $g\colon I\!R^n \times I\!R^m \to P$ be strictly differentiable at this point, and let the graph of $Q\colon I\!R^n \times I\!R^m \rightrightarrows P$ be locally closed around $(\bar{x}, \bar{y}, \bar{p})$ with $\bar{p} := -q(\bar{x}, \bar{y})$. Assume in addition that:*

(a) *either φ is strictly differentiable at $(\bar{x}, \bar{y})$, P is Banach, $Q = Q(y)$, and the partial derivative $\nabla_x q(\bar{x}, \bar{y})$ is surjective;*
(b) *or φ is lower regular at $(\bar{x}, \bar{y})$, P is Asplund, $Q = Q(x, y)$ is SNC and N-regular at $(\bar{x}, \bar{y}, \bar{p})$, and the adjoint generalized equation (23) has only the trivial solution $p^* = 0$.*

Then $(\bar{x}, \bar{y})$ is linearly subminimal with respect to (φ, S) in the MPEC formulated in (28), (22) if and only if there is $p^* \in P^*$ satisfying

$$0 \in \partial\varphi(\bar{x}, \bar{y}) + \nabla q(\bar{x}, \bar{y})^* p^* + D^*_N Q(\bar{x}, \bar{y}, \bar{p})(p^*). \tag{29}$$

Proof By definition, the linear subminimality of $(\bar{x}, \bar{y})$ with respect to (φ, S) in the MPEC formulated in (28), (22) means that $(\bar{x}, \bar{y})$ is linearly subminimal with respect to $(\varphi, \text{gph}\, S)$ in the sense studied in Theorem 5.2. Thus employing assertion (ii) of the latter theorem in case (a), we have that the condition

$$0 \in \nabla\varphi(\bar{x}, \bar{y}) + N\big((\bar{x}, \bar{y}); \text{gph}\, S\big) \tag{30}$$

is necessary and sufficient for the linear subminimality of $(\bar{x}, \bar{y})$ with respect to (φ, S) provided that φ is *strictly differentiable* at $(\bar{x}, \bar{y})$ with *no regularity* requirement on gph S. If φ is assumed to be *lower regular* at $(\bar{x}, \bar{y})$, then we employ Theorem 5.2(ii) in case (b) and conclude that the condition

$$0 \in \partial\varphi(\bar{x}, \bar{y}) + N\big((\bar{x}, \bar{y}); \text{gph}\, S\big), \tag{31}$$

is necessary and sufficient the linear subminimality of $(\bar{x}, \bar{y})$ with respect to (φ, S) *provided* that the graph of S is normally regular at $(\bar{x}, \bar{y})$; observe that the qualification condition (26) holds automatically by the Lipschitz continuity of φ around $(\bar{x}, \bar{y})$.

The rest of the proof of this theorem follows the one given for Theorem 4.6, where the normal cone $N\big((\bar{x}, \bar{y}); \text{gph}\, S)$ is computed and the N-regularity of S is justified by employing the results of [12, Theorem 4.44] under the assumptions on (P, q, Q) made in (a) and (b), respectively. In this way we derive criterion (31) from those in (29) and (30) and thus complete the proof of the theorem. $\square$

Finally, let us present specifications of the results of Theorem 5.3 in both cases (24) of MPECs with *composite subdifferential structures*, which are the most important for applications. In these case, we need to compute the normal coderivative $D^*_N Q(\bar{x}, \bar{y}, \bar{p})$ in (29) via the given data ψ and g of (24). To proceed, we will use the concept of the second-order subdifferential for extended-real-valued functions together with appropriate results (chain rules) of the second-order subdifferential calculus.

Given an extended-real-valued function $\varphi \colon X \to \overline{I\!R}$ finite at $\bar{x}$ and a basic first-order subgradient $\bar{y} \in \partial\varphi(\bar{x})$ from (10), recall [11,12] that the *second-order subdifferential* of φ at $\bar{x}$ relative to $\bar{y}$ is the mapping $\partial^2\varphi(\bar{x}, \bar{y}) \colon X^{**} \rightrightarrows X^*$ with the values

$$\partial^2\varphi(\bar{x}, \bar{y})(u) := (D^*_N \partial\varphi)(\bar{x}, \bar{y})(u), \quad u \in X^{**}, \tag{32}$$

i.e., it is defined as the (normal) coderivative of the first-order subdifferential mapping (an extension of the classical derivative-of-derivative approach in the second-order differentiation). If $\varphi \in C^2$ near $\bar{x}$, we have

$$\partial^2\varphi(\bar{x})(u) = \big\{\nabla^2\varphi(\bar{x})^* u\big\} \text{ for all } u \in X^{**}.$$

In the books [12,13], the reader can find a developed theory and various applications of the second-order subdifferential construction (32) as well as of its "mixed" counterpart (defined via the mixed coderivative) not used in this paper.

First we consider a specification of Theorem 5.3 in the case of $Q = \partial(\psi \circ g)$, i.e., when the field of the generalized equation under consideration is given in the *subdifferential form* with the so-called *composite potential*. As discussed in [10,12,15], such a model covers classical variational inequalities and their extensions. To obtain characterizations of linear subminimality for MPECs of this type, we involve second-order subdifferential chain rules giving a representation of $D^* Q = \partial^2(\psi \circ g)$ via the initial data (ψ, g). Again, we may apply only those calculus results that ensure chain rules as *equalities*. Since N-regularity does not seem to be a realistic property for subdifferential mappings with nonsmooth potentials, we restrict ourselves to case (a) of Theorem 5.3 combined with the coderivative calculation in [12, Theorem 4.49] for solution maps to parametric variational systems.

Corollary 3 (characterizing linear suboptimality for MPECs with composite potentials) *Let* $Q(y) = \partial(\psi \circ g)(y)$ *under the assumptions imposed in case* (a) *of Theorem* 5.3*, where*

$$S(x) := \big\{y \in I\!R^m \big|\, 0 \in q(x, y) + \partial(\psi \circ g)(y)\big\}, \tag{33}$$

where $q\colon I\!R^n \times I\!R^m \to I\!R^m$, $g\colon I\!R^m \to W$, $\psi\colon W \to \overline{I\!R}$, *and where* W *is Banach. Suppose in addition that* $g \in C^1$ *with the surjective derivative* $\nabla g(\bar{y})$*, that* $\nabla g(\cdot)$ *is strictly differentiable at* $\bar{y}$*, and that the graph of* $\partial\psi$ *is locally closed around* $(\bar{w}, \bar{v})$*, where* $\bar{w} := g(\bar{y})$ *and where* $\bar{v} \in W^*$ *is a unique functional satisfying*

$$-q(\bar{x}, \bar{y}) = \nabla g(\bar{y})^*\bar{v};$$

note that the closed-graph property of $\partial\psi$ *is automatic if* ψ *is either continuous or amenable around the reference point.*

Then $(\bar{x}, \bar{y})$ *is linearly suboptimal with respect to* (φ, S) *in the MPEC formulated in* (5.4)*,* (5.9) *if and only if the vector* $u \in I\!R^m$ *uniquely defined by the equation*

$$-\nabla_x\varphi(\bar{x}, \bar{y}) = \nabla_x q(\bar{x}, \bar{y})^* u$$

satisfies the relationship

$$0 \in \nabla_y\varphi(\bar{x}, \bar{y}) + \nabla_y q(\bar{x}, \bar{y})^* u + \nabla g(\bar{y})^*\partial^2\psi(\bar{w}, \bar{v})\big(\nabla g(\bar{y})u\big).$$

Proof Follows from Theorem 5.3(a) due to the calculation given in [12, Theorem 4.49] of the coderivative $D^*S(\bar{x}, \bar{y})$ for the mapping S defined in (33). This calculation is based on the second-order subdifferential chain rule from [12, Theorem 1.127]. □

The last corollary of Theorem 5.3(a) gives a specification of its result in the case of the second composite subdifferential structure in (24), which covers, e.g., the so-called *implicit complementarity problems*; see [10,12,15].

Corollary 4 (linear suboptimality for MPECs with composite subdifferential fields) *Let* $Q(y) = (\partial\psi \circ g)(y)$ *under the assumptions in case* (a) *of Theorem* 5.3*, where* $P = W^*$ *for some Banach space* W*, where*

$$S(x) := \big\{y \in I\!R^m \big|\, 0 \in q(x, y) + (\partial\psi \circ g)(y)\big\} \tag{34}$$

with $g\colon I\!R^m \to W$ *and* $\psi\colon W \to \overline{I\!R}$*, and where* g *is strictly differentiable at* $\bar{y}$ *with the surjective derivative* $\nabla g(\bar{y})$*. Denoting* $\bar{w} := g(\bar{y})$ *and* $\bar{p} := -q(\bar{x}, \bar{y})$*, we assume that the graph of* $\partial\psi$ *is locally closed around* $(\bar{w}, \bar{p})$*, which is automatic when* ψ *is either continuous or amenable. Then* $(\bar{x}, \bar{y})$ *is linearly suboptimal with respect to* (φ, S) *in the MPEC formulated in* (5.4)*,* (5.10) *if and only if the linear functional* $u \in W^{**}$ *uniquely defined by the equation*

$$-\nabla_x\varphi(\bar{x}, \bar{y}) = \nabla_x q(\bar{x}, \bar{y})^* u$$

satisfies the relationship

$$0 \in \nabla_y\varphi(\bar{x}, \bar{y}) + \nabla_y q(\bar{x}, \bar{y})^* u + \nabla g(\bar{y})^*\partial^2\psi(\bar{w}, \bar{p})(u).$$

Proof Follows from Theorem 5.3(a) due to the calculation given in [12, Proposition 4.53] of the coderivative $D^*S(\bar{x}, \bar{y})$ for the mapping S defined in (34). This calculation is based on the second-order subdifferential chain rule from [12, Theorem 1.66]. □

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